# Dispersionful Version of WDVV Associativity System

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**Abstract**

B.A. Dubrovin proved that remarkable WDVV associativity equations are integrable systems. In a simplest nontrivial three-component case these equations can be written as a nondiagonalizable hydrodynamic type system equivalent to a symmetric reduction of the three wave interaction and to the matrix Hopf equation. Then E.V. Ferapontov and O.I. Mokhov found a local Hamiltonian structure. Finally E.V. Ferapontov, C.A.P. Galvão, O.I. Mokhov, Ya. Nutku found a second local Hamiltonian structure. Both local Hamiltonian structure are homogeneous of first and third order (respectively) of Dubrovin–Novikov type.

In our paper we suggest a special scaling procedure for independent variables applicable for homogeneous nonlinear PDE's, which allows to incorporate an auxiliary parameter $\epsilon$, such that a corresponding "intermediate" system possesses two remarkable limits: a high-frequency limit ($\epsilon \to \infty$) back to the original system and a dispersionless limit ($\epsilon \to 0$) which yields diagonalizable integrable hydrodynamic type system. This means that our procedure allows to transform a homogeneous third order local Hamiltonian structure to non-homogeneous of third order. Thus we create an integrable hierarchy equipped by a pair of local Hamiltonian structures, which (both of them) possess a dispersionless limit. Also we show that this bi-Hamiltonian diagonalizable hydrodynamic type system possesses at least two different dispersive integrable extensions (in a framework of B.A. Dubrovin's approach).

*On the occasion of Boris Dubrovin's 65th birthday*

# Contents



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# 1 Introduction

Plenty of integrable systems possess a dispersionless limit, which is a semi-Hamiltonian hydrodynamic type system. Most well-known examples are: the Korteweg de Vries equation, the Kaup–Boussinesq system, the Boussinesq equation. However some other integrable systems have no such a dispersionless limit (at least nobody knows how to derive it). Examples: the Bonnet equation (also known as the Sin-Gordon equation), the Krichever–Novikov equation, the Landau–Lifshitz equation.

In this paper we start from the hydrodynamic type system

$$a^1_y = a^2_z, \quad a^2_y = a^3_z, \quad a^3_y = [(a^2)^2 - a^1a^3]_z. \tag{1}$$

This system is nondiagonalizable, i.e. is not semi-Hamiltonian, but is integrable by the inverse scattering transform. This system was derived as a single equation ($a^1 = f_{zzz}, a^2 = f_{yzz}, a^3 = f_{yyz}$)

$$f_{yyy} = f^2_{yzz} - f_{zzz}f_{yyz} \tag{2}$$

by B.A. Dubrovin (see detail in [1]) and rewritten in the above hydrodynamic form by E.V. Ferapontov and O.I. Mokhov (see detail in [5]). A standard approach established by B.A. Dubrovin for reconstruction of higher dispersive corrections to this case is inapplicable, because a dispersionless limit must be a semi-Hamiltonian hydrodynamic type system. Nevertheless we were able to construct a new integrable system, which we call the "intermediate" dispersive system. This intermediate dispersive system essentially depends on an extra parameter $\epsilon$ such that: if $\epsilon \to \infty$, in this high-frequency limit one can obtain the above non-diagonalizable hydrodynamic type system; if $\epsilon \to 0$, in this dispersionless limit one can obtain a semi-Hamiltonian hydrodynamic type system.

The main advantage of our construction is that we just re-scaled the independent variable $z \to z(x, \epsilon)$ in a linear equation of third order, which is a first "half" of the Lax pair associated with the above non-diagonalizable hydrodynamic type system. This non-diagonalizable hydrodynamic type system possesses a bi-Hamiltonian structure (see detail in [4]). In this paper we show that the intermediate dispersive system also has a bi-Hamiltonian structure as well as its dispersionless limit.

The structure of the paper is as follows. In Section 2 we briefly describe a bi-Hamiltonian formulation of the WDVV associativity system and discuss a Theorem about a relationship between flat coordinates and a momentum for a local Hamiltonian structure of Dubrovin–Novikov type of first order [2]. In Section 3 we construct a one-parametric transformation from the first "half" of the Lax pair (determining the WDVV associativity system) to another first "half" of a Lax pair, which determines a new integrable dispersive system, which depends on an arbitrary parameter. In Section 4 we present a procedure for construction of local Hamiltonian structures of Dubrovin–Novikov type. In Subsection 4.1 we expand a generating function of conservation law densities and extract two quadratic relationships between conservation law densities, which should be identified as flat coordinates and momenta. In Subsection 4.2 we find a non-evolutionary compact form for the intermediate dispersive system, which allows to consider a high-frequency limit. We construct a bi-Hamiltonian structure and present two corresponding local Lagrangian representations. In Subsection 4.3 we demonstrate that a high-frequency limit of this intermediate dispersive system is precisely the WDVV associativity system (1). In Section

5 we slightly re-scale the independent variable "$x$" and the spectral parameter $\lambda$. Then all conservation law densities become of hydrodynamic type in a dispersionless limit. In Subsection 5.1 we consider a semi-Hamiltonian system, which follows from a dispersionless limit. We construct its bi-Hamiltonian structure and show that this dispersionless system coincides with a dispersionless limit of the remarkable Yajima–Oikawa system. Thus this bi-Hamiltonian hydrodynamic type system possesses at least two different dispersive integrable extensions. In Subsection 5.2 we find a non-evolutionary compact form for the intermediate dispersive system, which allows to consider a dispersionless limit. In Section 6 we briefly consider the Yajima–Oikawa system, its bi-Hamiltonian structure and a dispersionless limit. In Section 7 we discuss a similar transformation for the second "half" of the Lax pair (determining the WDVV associativity system). Finally in Conclusion

## 2 Bi-Hamiltonian Structure of WDVV Associativity System

The remarkable WDVV associativity equation (2) written in an equivalent hydrodynamic form (1) admits a bi-Hamiltonian structure (see detail in [4])

$$a^i_y = A_1^{ij}\frac{\delta \mathbf{H}_2}{\delta a^j} = A_2^{ij}\frac{\delta \mathbf{H}_1}{\delta a^j},$$

where two compatible local Hamiltonian operators $\hat{A}_1$ and $\hat{A}_2$ are

$$\hat{A}_1 = \begin{pmatrix} -\frac{3}{2}\partial_z & \frac{1}{2}\partial_z a^1 & \partial_z a^2 \\ \frac{1}{2}a^1\partial_z & \frac{1}{2}(\partial_z a^2 + a^2\partial_z) & \frac{3}{2}a^3\partial_z + a^3_z \\ a^2\partial_z & \frac{3}{2}\partial_z a^3 - a^3_z & [(a^2)^2 - a^1a^3]\partial_z + \partial_z[(a^2)^2 - a^1a^3] \end{pmatrix}, \tag{3}$$

$$\hat{A}_2 = \begin{pmatrix} 0 & 0 & \partial_z^3 \\ 0 & \partial_z^3 & -\partial_z^2 a^1\partial_z \\ \partial_z^3 & -\partial_z a^1\partial_z^2 & \partial_z^2 a^2\partial_z + \partial_z a^2\partial_z^2 + \partial_z a^1\partial_z a^1\partial_z \end{pmatrix} \tag{4}$$

and Hamiltonian densities, respectively, are $h_2 = a^3$, $h_1 = -\frac{1}{2}a^1(\partial_z^{-1}a^2)^2 - (\partial_z^{-1}a^2)(\partial_z^{-1}a^3)$, where $H_i = \int h_i dz$. These two Hamiltonian operators $\hat{A}_1$ and $\hat{A}_2$ are homogeneous (see [2, 3] for more details).

In this paper we introduce new integrable hierarchy, which contains the above three-component system as a so called "high frequency" limit. We prove that this intermediate dispersive system also is bi-Hamiltonian. In this paper we use the following

**Theorem** (see detail in [10]): *If hydrodynamic type system written in a conservative form*

$$u^k_t = (f^k(\mathbf{u}))_x \tag{5}$$

*also possesses an extra conservation law $\partial_t p(\mathbf{u}) = \partial_x g(\mathbf{u})$ such that (here $g_{km}$ is a constant symmetric nondegenerate matrix)*

$$p = \frac{1}{2}g_{km}u^k u^m, \tag{6}$$

*then $u^k$ are flat coordinates, $p(\mathbf{u})$ is a momentum density, and this hydrodynamic type system has a local Hamiltonian structure*

$$u^i_t = g^{ik}\left(\frac{\partial h}{\partial u^k}\right)_x. \tag{7}$$

**Proof**: The conservation law $\partial_t p(\mathbf{u}) = \partial_x g(\mathbf{u})$ leads to

$$\frac{\partial p}{\partial u^i}\frac{\partial f^i}{\partial u^k}u^k_x = \frac{\partial g}{\partial u^k}u^k_x.$$

Then (see (6))

$$\frac{\partial g}{\partial u^k} = g_{sm}u^m\frac{\partial f^s}{\partial u^k} = \frac{\partial}{\partial u^k}(g_{sm}u^m f^s) - g_{km}f^m.$$

This means that the last term $g_{km}f^m$ must be a partial derivative of some function $h(\mathbf{u})$. Then indeed $f^i = g^{ik}\frac{\partial h}{\partial u^k}$ (cf. (5) and (7)).

This theorem can be expand to dispersive integrable systems. *If a conservative dispersive system* (cf. (5))

$$u^k_t = (f^k(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...))_x \tag{8}$$

*also has an extra conservation law $\partial_t p(\mathbf{u}) = \partial_x g(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...)$ such that $p(\mathbf{u})$ is determined by (6), then (8) has a local Hamiltonian structure.* Indeed (see the above Theorem)

$$\partial_x g(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...) = \frac{\partial p}{\partial u^k}(f^k(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...))_x = g_{km}u^m(f^k(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...))_x$$

$$= (g_{km}u^m f^k(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...))_x - g_{km}f^k(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...)u^m_x.$$

Now we utilize a well-known relationship (see, for instance, [7])

$$\frac{\delta \mathbf{H}}{\delta u^k}u^k_x = (Q(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...))_x.$$

Thus a function $h(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...)$ exists if

$$g_{km}f^m(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...) = \frac{\delta \mathbf{H}}{\delta u^k},$$

where $\mathbf{H} = \int h(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...)dx$. This means that dispersive system (8) takes the Hamiltonian form

$$u^k_t = g^{km}\left(\frac{\delta \mathbf{H}}{\delta u^m}\right)_x.$$

The operator $\hat{A}_1$ was found in [5], and it is completely specified by its leading term, which is a contravariant flat pseudo-Riemannian metric $g^{ik}$. The observation that led to finding $\hat{A}_1$ was that the eigenvalues $u^k(\mathbf{a})$ of the matrix $\mathbf{B}$ are conservation law densities in the Lax pair of the system (1)

$$\boldsymbol{\psi}_z = \lambda\mathbf{B}\boldsymbol{\psi} = \lambda\begin{pmatrix} 0 & 1 & 0 \\ a^2 & a^1 & 1 \\ a^3 & a^2 & 0 \end{pmatrix}\boldsymbol{\psi}, \qquad \boldsymbol{\psi}_y = \lambda\mathbf{C}\boldsymbol{\psi} = \lambda\begin{pmatrix} 0 & 0 & 1 \\ a^3 & a^2 & 0 \\ (a^2)^2 - a^1a^3 & a^3 & 0 \end{pmatrix}\boldsymbol{\psi}. \tag{9}$$

If the system is rewritten using the above eigenvalues as new dependent variables $u^k$, i.e., using the point transformation

$$a^1 = u^1 + u^2 + u^3, \quad a^2 = -\frac{1}{2}(u^1u^2 + u^1u^3 + u^2u^3), \quad a^3 = u^1u^2u^3, \tag{10}$$

the operator $\hat{A}_1$ becomes evident and is of the type $A_1^{ij} = K^{ij}\partial_x$, where

$$K = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \tag{11}$$

and the Hamiltonian density is $h_2 = u^1u^2u^3$. In these new coordinates WDVV associativity system (1) takes the form

$$u_y^1 = \frac{1}{2}(u^2u^3 - u^1u^2 - u^1u^3)_z, \quad u_y^2 = \frac{1}{2}(u^1u^3 - u^1u^2 - u^2u^3)_z, \quad u_y^3 = \frac{1}{2}(u^1u^2 - u^1u^3 - u^2u^3)_z. \tag{12}$$

The quadratic relationship (see (10))

$$a^2 = -\frac{1}{2}(u^1u^2 + u^1u^3 + u^2u^3) \tag{13}$$

is nothing but (6) for the WDVV associativity system (1), where $g^{ij} = K^{ij}$.

# 3 Transformation to the Dispersionful Version

In this Section we consider the left matrix equation of linear problem (9)

$$\boldsymbol{\psi}_z = \lambda \begin{pmatrix} 0 & 1 & 0 \\ a^2 & a^1 & 1 \\ a^3 & a^2 & 0 \end{pmatrix} \boldsymbol{\psi}.$$

Instead of three linear equations

$$\psi_z = \lambda\psi_1, \quad \psi_{1,z} = \lambda a^2\psi + \lambda a^1\psi_1 + \lambda\psi_2, \quad \psi_{2,z} = \lambda a^3\psi + \lambda a^2\psi_1,$$

we shall use a single scalar ordinary equation of third order

$$\psi_{zzz} = \lambda a^1\psi_{zz} + (2\lambda^2a^2 + \lambda a_z^1)\psi_z + (\lambda^2a_z^2 + \lambda^3a^3)\psi, \tag{14}$$

where $\psi_1 = \lambda^{-1}\psi_z, \psi_2 = \lambda^{-2}\psi_{zz} - \lambda^{-1}a^1\psi_z - a^2\psi$.

Now we introduce the transformation

$$z = \epsilon(e^{x/\epsilon} - 1), \tag{15}$$

where $\epsilon$ is an arbitrary parameter. Then $\partial_z \to e^{-x/\epsilon}\partial_x$. If $\epsilon \to \infty$, then $z(x, \epsilon) \to x$. In this case (14) takes the form

$$\psi_{xxx} - \left(\frac{3}{\epsilon} + \lambda w^1\right)\psi_{xx} \tag{16}$$

$$+\left[\frac{2}{\epsilon^2}+\lambda\left(\frac{2}{\epsilon}w^1-w^1_x\right)-2\lambda^2w^2\right]\psi_x+\left[\lambda^2\left(\frac{2}{\epsilon}w^2-w^2_x\right)-\lambda^3w^3\right]\psi=0,$$

where we introduced new field variables $w^k$ and $v^m$ instead of $a^k$ and $u^m$, respectively (see (1), (9), (12)):

$$a^1=w^1e^{-x/\epsilon},\quad a^2=w^2e^{-2x/\epsilon},\quad a^3=w^3e^{-3x/\epsilon},\quad u^k=v^ke^{-x/\epsilon},\ k=1,2,3.$$

So variable coefficients of third order differential equation (16) do not depend explicitly on independent variable "$x$" as well as in (14). This equation (16) also can be written as a linear problem (9) in the matrix form

$$\tilde{\boldsymbol{\psi}}_x=\left[\lambda\begin{pmatrix}0&1&0\\w^2&w^1&1\\w^3&w^2&0\end{pmatrix}+\frac{1}{\epsilon}\begin{pmatrix}0&0&0\\0&1&0\\0&0&2\end{pmatrix}\right]\tilde{\boldsymbol{\psi}},\tag{17}$$

where (the column of the vector function $\tilde{\boldsymbol{\psi}}$ contains three components $\tilde{\psi}=\psi,\tilde{\psi}_1$ and $\tilde{\psi}_2$, consequently)

$$\tilde{\psi}_1=\frac{1}{\lambda}\psi_x,\quad \tilde{\psi}_2=\frac{1}{\lambda^2}\psi_{xx}-\frac{1}{\lambda}\left(\frac{1}{\epsilon\lambda}+w^1\right)\psi_x-w^2\psi.$$

Below in this Section we consider construction of an integrable hierarchy dependent on the extra parameter $\epsilon$.

# 4 The Intermediate Integrable Hierarchy

In comparison with a standard construction of integrable systems based on a compatibility condition of a corresponding Lax pair, in this Section we follow to another strategy:

1. We compute first conservation law densities from a generating function of conservation law densities;

2. We select quadratic relationships of type (6);

3. We consider the conservation law densities involved in these quadratic relationships as flat coordinates and momenta, we can take any other conservation law densities as Hamiltonian densities to create corresponding Hamiltonian systems;

4. A compatible pair of Hamiltonian operators must determine each member of an integrable hierarchy by an appropriate choice of corresponding pairs of Hamiltonian densities. We present such a pair of Hamiltonian densities for the intermediate dispersive system.

## 4.1 A Generating Function of Conservation Law Densities

The substitution

$$\psi=e^{\int rdx}\tag{18}$$

into third order linear differential equation (16) leads to the second order ordinary nonlinear differential equation

$$r_{xx}+3rr_x+r^3-\left(\frac{3}{\epsilon}+\lambda w^1\right)(r_x+r^2)\tag{19}$$

$$+\left[\frac{2}{\epsilon^2}+\lambda\left(\frac{2}{\epsilon}w^1-w_x^1\right)-2\lambda^2w^2\right]r+\lambda^2\left(\frac{2}{\epsilon}w^2-w_x^2-\lambda w^3\right)=0,$$

where the function $r$ is a generating function of conservation law densities. Indeed, quasipolynomial conservation law densities can be obtained by substitution of the Laurent series ($\lambda\rightarrow\infty$)

$$r=\lambda a_{-1}+a_0+\frac{a_1}{\lambda}+\frac{a_2}{\lambda}+\frac{a_3}{\lambda^3}+...,\tag{20}$$

where all coefficients $a_k(\mathbf{w},\mathbf{w}_x,\mathbf{w}_{xx},...)$ can be found iteratively:

$$(a_{-1})^3-(a_{-1})^2w^1-2a_{-1}w^2-w^3=0,\tag{21}$$

$$[3(a_{-1})^2-2a_{-1}w^1-2w^2]\left(\frac{1}{\epsilon}-a_0\right)=(3a_{-1}-w^1)a_{-1,x}-a_{-1}w_x^1-w_x^2,\tag{22}$$

$$[3(a_{-1})^2-2a_{-1}w^1-2w^2]a_1+\left(\frac{2}{\epsilon^2}-\frac{6}{\epsilon}a_0+3(a_0)^2\right)a_{-1}+\left(\frac{2}{\epsilon}-a_0\right)a_0w^1\tag{23}$$

$$+(3a_{-1}-w^1)a_{0,x}+3\left(a_0-\frac{1}{\epsilon}\right)a_{-1,x}-a_0w_x^1+a_{-1,xx}=0,...$$

The first equation (21) determines three roots $v^m$, if one assumes that functions $w^k$ are given. Or we can consider these roots $v^m$ as given functions, then functions $w^k$ are polynomials according to the Viète theorem, i.e. (cf. (10))

$$w^1=v^1+v^2+v^3,\quad w^2=-\frac{1}{2}(v^1v^2+v^1v^3+v^2v^3),\quad w^3=v^1v^2v^3.\tag{24}$$

Existence of these three roots $v^m$ means existence of three branches of conservation law densities (see (20))

$$r^{(k)}=\lambda v^k+a_0^{(k)}+\frac{a_1^{(k)}}{\lambda}+\frac{a_2^{(k)}}{\lambda}+\frac{a_3^{(k)}}{\lambda^3}+...,\quad k=1,2,3,\tag{25}$$

where all other higher conservation law densities $a_k^{(m)}$ can be expressed as rational functions with respect to three roots $v^k$ and their higher derivatives. For instance (see (22) and (23), respectively)

$$a_0^{(1)}=\frac{1}{\epsilon}-\frac{1}{2}\partial_x\ln[(v^1-v^2)(v^1-v^3)],$$

$$a_1^{(1)}=\frac{(\frac{2}{\epsilon}-\partial_x)\left[(2v^1-v^2-v^3)a_0^{(1)}+(\partial_x-\frac{1}{\epsilon})v^1\right]-(2v^1-v^2-v^3)\left(a_0^{(1)}\right)^2}{(v^1-v^2)(v^1-v^3)}$$

$$a_2^{(1)}=\frac{(\partial_x-\frac{2}{\epsilon})\left[\frac{3}{2}\left(a_0^{(1)}\right)^2+(\partial_x-\frac{1}{\epsilon})a_0^{(1)}+(2v^1-v^2-v^3)a_1^{(1)}\right]+\left(a_0^{(1)}\right)^3+2(2v^1-v^2-v^3)a_0^{(1)}a_1^{(1)}}{(v^1-v^2)(v^1-v^3)}.$$

Lower (nonlocal) conservation law densities can be found by substitution another expansion ($\lambda\rightarrow 0$)

$$r=\lambda^2(q_0+\lambda q_1+\lambda^2q_2+\lambda^3q_3+...),\tag{26}$$

into (19). Corresponding conservation law densities $q_k$ cannot be expressed via roots $v^k$ and their finite number of derivatives, i.e.

$$\left(\partial_x - \frac{1}{\epsilon}\right) q_0 = w^2, \tag{27}$$

$$\left(\partial_x - \frac{1}{\epsilon}\right)\left(\partial_x - \frac{2}{\epsilon}\right) q_1 = w^3 - \frac{2}{\epsilon} q_0 w^1 + (q_0 w^1)_x, \tag{28}$$

$$\left(\partial_x - \frac{1}{\epsilon}\right)\left(\partial_x - \frac{2}{\epsilon}\right) q_2 = \frac{3}{\epsilon}(q_0)^2 - \frac{2}{\epsilon} q_1 w^1 + 2q_0 w^2 + \left(q_1 w^1 - \frac{3}{2}(q_0)^2\right)_x, \tag{29}$$

$$\left(\partial_x - \frac{1}{\epsilon}\right)\left(\partial_x - \frac{2}{\epsilon}\right) q_3 = (q_0)^2 w^1 + 2q_1 w^2 + \left(\partial_x - \frac{2}{\epsilon}\right)(q_2 w^1) - 3\left(\partial_x - \frac{2}{\epsilon}\right)(q_1 q_0), \ldots \tag{30}$$

However some important observations one can made analyzing this expansion and Viète's formulas (24). Indeed:

1. $w^1$ is a conservation law density, because $w^1$ is a linear combination of roots $v^k$, which are conservation law density according to the construction (i.e. they are first elements of expansions (25) of the generation function of conservation law densities $r$);

2. the first relationship (27) means that $w^2$ is a conservation law density, because $q_0$ is a conservation law density and any conservation law density is determined up to total derivatives;

3. thus the central quadratic relationship in (24) means that any integrable system associated with (16) or (17) can possess a *first* local Hamiltonian structure (see (6) and (7));

4. the second relationship (28) means that the expression $w^3 - \frac{2}{\epsilon} q_0 w^1$ is a conservation law density, because $q_1$ is a conservation law density and any conservation law density is determined up to total derivatives;

5. the third relationship (29) means that any integrable system associated with (16) can possess a *second* local Hamiltonian structure, because (see (6) and (7)) $q_2$ is a quadratic expression in terms of functions $w^1, q_0, q_1$ (all these four functions $w^1, q_0, q_1, q_2$ are conservation law densities) up to a total derivative:

$$-\frac{1}{\epsilon} q_2 + (q_2)_x = q_1 w^1 - \frac{1}{2}(q_0)^2, \tag{31}$$

Here we took into account that the conservation law density $w^2$ can be expressed via $q_0$ from (27).

## 4.2 A Non-Evolutionary Form

In this Subsection we are going to construct an integrable system whose the so called high-frequency limit ($\epsilon \to \infty$) is precisely original WDVV associativity system (1). In such a case, this dispersive integrable system can be written, for instance, via flat coordinates $v^k$. Its Hamiltonian density is $w^3 - \frac{2}{\epsilon} q_0 w^1$, because the Hamiltonian density of (1) is $w^3$ (see

[4]). However, taking into account that (see (27)) the relationship between field variables $q_0$ and $w^2$ is uninvertible in a local sense, the Hamiltonian density becomes nonlocal:

$$h_2 = v^1v^2v^3 + \frac{1}{\epsilon}(v^1+v^2+v^3)\left(\partial_x - \frac{1}{\epsilon}\right)^{-1}(v^1v^2+v^1v^3+v^2v^3). \tag{32}$$

Expanding in $\epsilon$, this Hamiltonian density reads ($\epsilon \to \infty$)

$$h_2 = v^1v^2v^3 + \frac{1}{\epsilon}(v^1+v^2+v^3)\partial_x^{-1}\left(1+\frac{1}{\epsilon}\partial_x^{-1}+\frac{1}{\epsilon^2}\partial_x^{-2}+\frac{1}{\epsilon^3}\partial_x^{-3}+...\right)(v^1v^2+v^1v^3+v^2v^3).$$

Under the differential substitutions

$$v^k = \left(\partial_x + \frac{1}{\epsilon}\right)\eta^k, \tag{33}$$

Hamiltonian density (32) reduces to the local form

$$h_2 = v^1v^2v^3 - \frac{1}{\epsilon}(v^1v^2+v^1v^3+v^2v^3)(\eta^1+\eta^2+\eta^3),$$

while local Hamiltonian structure (see (11)) becomes nonlocal:

$$\eta_t^i = -K^{im}\left(\partial_x^2 - \frac{1}{\epsilon^2}\right)^{-1}\left(\frac{\delta \mathbf{H}_2}{\delta \eta^m}\right)_x.$$

This means that the intermediate dispersive system has a non-evolutionary form

$$\frac{1}{\epsilon^2}\eta_t^i - \eta_{xxt}^i = K^{im}\left(\frac{\delta \mathbf{H}_2}{\delta \eta^m}\right)_x.$$

**Remark**: The intermediate dispersive system also has a local Lagrangian representation

$$S_2 = \int\left[\frac{1}{2}K_{im}\left(\frac{1}{\epsilon^2}\phi_x^i - \phi_{xxx}^i\right)\phi_t^m - h_2\right]dxdt, \tag{34}$$

where $\phi^k$ are potentials such that $\eta^k = \phi_x^k$ and $K_{ij}$ is determined by (11).

Also independently taking into account (31) the intermediate dispersive system can be written in the second Hamiltonian form

$$w_t^1 = \left(\frac{\delta \mathbf{H}_1}{\delta q_1}\right)_x, \quad q_{0,t} = -\left(\frac{\delta \mathbf{H}_1}{\delta q_0}\right)_x, \quad q_{1,t} = \left(\frac{\delta \mathbf{H}_1}{\delta w^1}\right)_x,$$

where (see (30))

$$\mathbf{H}_1 = \int\left[\frac{1}{2}(q_0)^2\eta_x + q_1q_{0,x} + \frac{1}{\epsilon}q_1\left(2q_0 + \eta\eta_x + \frac{1}{\epsilon}\eta^2\right)\right]dx,$$

where we introduced the field variable $\eta$ such that (see (33) and the first relationship in (24))

$$w^1 = \left(\partial_x + \frac{1}{\epsilon}\right)\eta \quad \leftrightarrow \quad \eta = \eta^1+\eta^2+\eta^3.$$

In field variables $q_0, q_1, \eta$ the intermediate dispersive system takes the non-evolutionary form

$$\left(\partial_x + \frac{1}{\epsilon}\right)\eta_t = \left(q_{0,x} + \frac{1}{\epsilon}(2q_0 + \eta\eta_x) + \frac{1}{\epsilon^2}\eta^2\right)_x, \quad q_{0,t} = \left(q_{1,x} - q_0\eta_x - \frac{2}{\epsilon}q_1\right)_x, \tag{35}$$

$$\left(\partial_x - \frac{1}{\epsilon}\right)q_{1,t} = \left(\frac{1}{2}[(q_0)^2]_x + \frac{1}{\epsilon}\eta q_{1,x} - \frac{2}{\epsilon^2}\eta q_1\right)_x,$$

where above Hamiltonian structure becomes nonlocal:

$$\eta_t = \left(\partial_x + \frac{1}{\epsilon}\right)^{-1}\left(\frac{\delta \mathbf{H}_1}{\delta q_1}\right)_x, \quad q_{0,t} = -\left(\frac{\delta \mathbf{H}_1}{\delta q_0}\right)_x, \quad q_{1,t} = -\left(\partial_x - \frac{1}{\epsilon}\right)^{-1}\left(\frac{\delta \mathbf{H}_1}{\delta \eta}\right)_x.$$

Under the potential substitutions

$$\eta = \left(\partial_x - \frac{1}{\epsilon}\right)\zeta_x, \quad q_0 = Q_{0,x}, \quad q_1 = Q_{1,x}$$

we obtain another local Lagrangian representation

$$S_1 = \int \left[\frac{1}{2}\left(\zeta_{xxx} - \frac{1}{\epsilon^2}\zeta_x\right)Q_{1,t} + \frac{1}{2}\left(Q_{1,xxx} - \frac{1}{\epsilon^2}Q_{1,x}\right)\zeta_t - \frac{1}{2}Q_{0,x}Q_{0,t} - h_1\right]dxdt. \tag{36}$$

### 4.3 The WDVV Associativity System as a High-Frequency Limit

Higher and lower conservation law densities for the WDVV associativity system (1) can be obtained (see transformation (15)) in the *high-frequency limit* $\epsilon \to \infty$. This means that expansion (25) leads to little bit more simple differential consequences:

$$a_0^{(1)} = -\frac{1}{2}(\ln(u^1 - u^2)(u^1 - u^3))_z,$$

$$a_1^{(1)} = -\frac{\left[(2u^1 - u^2 - u^3)a_0^{(1)} + u_z^1\right]_z + (2u^1 - u^2 - u^3)\left(a_0^{(1)}\right)^2}{(u^1 - u^2)(u^1 - u^3)},$$

$$a_2^{(1)} = \frac{\left[\frac{3}{2}\left(a_0^{(1)}\right)^2 + \left(a_0^{(1)}\right)_z + (2u^1 - u^2 - u^3)a_1^{(1)}\right]_z + \left(a_0^{(1)}\right)^3 + 2(2u^1 - u^2 - u^3)a_0^{(1)}a_1^{(1)}}{(u^1 - u^2)(u^1 - u^3)}, \ldots$$

where $u^k$ are roots (see (24)) of third order algebraic equation (21). They are nothing but flat coordinates of first Hamiltonian structure (see (3), (10), (11), (12)) of WDVV associativity system (1). Coefficients of expansion (26) significantly simplify (cf. (27), (28), (29)):

$$q_{0,z} = w^2, \tag{37}$$

$$q_{1,zz} = w^3 + (q_0 w^1)_z, \tag{38}$$

$$q_{2,zz} = 2q_0 w^2 + \left(q_1 w^1 - \frac{3}{2}(q_0)^2\right)_z, \tag{39}$$

$$q_{3,zz} = (q_2 w^1 - 3q_1 q_0)_z + (q_0)^2 w^1 + 2q_1 w^2, ... \tag{40}$$

Taking into account that $w^2 = q_{0,z}$, equation (39) can be integrated once:

$$q_{2,z} = q_1 w^1 - \frac{1}{2}(q_0)^2. \tag{41}$$

WDVV associativity system (1) takes the form

$$w^1_y = q_{0,zz}, \quad q_{0,y} = (q_{1,z} - q_0 w^1)_z, \quad q_{1,y} = \frac{1}{2}[(q_0)^2]_z, \tag{42}$$

where we took into account that $w^3 = (q_{1,z} - q_0 w^1)_z$ (see (38)).

Since $w^1, q_0, q_1$ are conservation law densities, quadratic relationship (41) ($q_{2,z}$ is a trivial conservation law density) again means that the high-frequency limit (i.e. WDVV associativity system and all its commuting flows) possesses a bi-Hamiltonian structure, as it was earlier proved in [4]). Thus evolutionary system (42) can be written in the Hamiltonian form

$$w^1_y = \left(\frac{\delta \mathbf{H}_1}{\delta q_1}\right)_z, \quad q_{0,y} = -\left(\frac{\delta \mathbf{H}_1}{\delta q_0}\right)_z, \quad q_{1,y} = \left(\frac{\delta \mathbf{H}_1}{\delta w^1}\right)_z, \tag{43}$$

where the Hamiltonian density $h_1 = q_1 q_{0,z} + \frac{1}{2}(q_0)^2 w^1$ (see (40), here we substituted $w^2$ from (37)). Under differential substitutions (37), (38) in the original field variables $w^k$ Hamiltonian structure (43) of evolutionary system (42) becomes precisely (4). This confirms that WDVV associativity system (1) possesses a bi-Hamiltonian structure.

# 5 A Dispersionless Limit of Intermediate Dispersive System

To derive a correct dispersionless limit ($\epsilon \to 0$) first we need to re-scale a spectral parameter $\lambda \to \Lambda/\epsilon$. Then substitution (cf. (18))

$$\psi = \exp\left(\frac{1}{\epsilon}\int r dx\right)$$

into (16) leads to the second order ordinary nonlinear differential equation (cf. (19))

$$\begin{gathered}\epsilon^2 r_{xx} + 3\epsilon r r_x + r^3 - \left(3 + \Lambda w^1\right)\left(\epsilon r_x + r^2\right) \\ + \left[2 + \Lambda\left(2w^1 - \epsilon w^1_x\right) - 2\Lambda^2 w^2\right] r + \Lambda^2\left(2w^2 - \epsilon w^2_x - \Lambda w^3\right) = 0.\end{gathered} \tag{44}$$

Then the first expansion (cf. (20))

$$r = \Lambda b_{-1} + b_0 + \frac{b_1}{\Lambda} + \frac{b_2}{\Lambda^2} + \frac{b_3}{\Lambda^3} + ..., \quad \Lambda \to \infty \tag{45}$$

yields infinitely many differential consequences, which slightly different in comparison with (21), (22), (23):

$$(b_{-1})^3 - (b_{-1})^2 w^1 - 2b_{-1} w^2 - w^3 = 0,$$

$$[3(b_{-1})^2 - 2b_{-1}w^1 - 2w^2](1-b_0) = \epsilon[(3b_{-1} - w^1)b_{-1,x} - b_{-1}w^1_x - w^2_x],$$
$$[3(b_{-1})^2 - 2b_{-1}w^1 - 2w^2]b_1 + [2 - 6b_0 + 3(b_0)^2]b_{-1} + (2-b_0)b_0w^1$$
$$+\epsilon[(3b_{-1} - w^1)b_{0,x} + 3(b_0 - 1)b_{-1,x} - b_0w^1_x] + \epsilon^2 b_{-1,xx} = 0, \ldots$$

The second expansion (cf. (26))

$$r = \Lambda^2(c_0 + \Lambda c_1 + \Lambda^2 c_2 + \Lambda^3 c_3 + \ldots), \quad \Lambda \to 0 \tag{46}$$

yields (cf. (27), (28), (29), (30)):

$$(1 - \epsilon\partial_x)c_0 = -w^2, \tag{47}$$

$$(1 - \epsilon\partial_x)(2 - \epsilon\partial_x)c_1 = w^3 - 2c_0w^1 + \epsilon(c_0w^1)_x, \tag{48}$$

$$(1 - \epsilon\partial_x)(2 - \epsilon\partial_x)c_2 = 3(c_0)^2 - 2c_1w^1 + 2c_0w^2 + \epsilon\left(c_1w^1 - \frac{3}{2}(c_0)^2\right)_x, \tag{49}$$

$$(1 - \epsilon\partial_x)(2 - \epsilon\partial_x)c_3 = (c_0)^2w^1 + 2c_1w^2 + (2 - \epsilon\partial_x)(3c_0c_1 - c_2w^1), \ldots \tag{50}$$

### 5.1 A Egorov bi-Hamiltonian Hydrodynamic Type System

In the dispersionless limit, the algebraic relationship

$$r^3 - \left(3 + \Lambda w^1\right) r^2 + 2(1 + \Lambda w^1 - \Lambda^2 w^2)r + 2\Lambda^2 w^2 - \Lambda^3 w^3 = 0 \tag{51}$$

follows from (44). Infinitely many *rational* conservation law densities can be obtained by substitution (45) into (51):

$$b_0 = 1, \quad b_1^{(1)} = -\frac{v^2 + v^3}{(v^1 - v^2)(v^1 - v^3)}, \quad b_2^{(1)} = 0,$$

$$b_3^{(1)} = \frac{(v^2 + v^3)\left((v^1)^2 - v^1(v^2 + v^3) + (v^2)^2 + (v^3)^2 + v^2v^3\right)}{(v^1 - v^2)^3(v^1 - v^3)^3}, \ldots$$

where $v^k$ are roots of the cubic algebraic equation (cf. (21))

$$(b_{-1})^3 - (b_{-1})^2w^1 - 2b_{-1}w^2 - w^3 = 0. \tag{52}$$

Infinitely many *polynomial* conservation law densities (cf. with the high-frequency limit: (37), (38), (39), (40)) can be obtained by substitution (46) into (51):

$$c_0 = -w^2, \tag{53}$$

$$c_1 = \frac{1}{2}w^3 - c_0w^1,$$
$$c_2 = \frac{3}{2}(c_0)^2 - c_1w^1 + c_0w^2, \tag{54}$$
$$c_3 = \frac{1}{2}(c_0)^2w^1 + c_1w^2 + 3c_0c_1 - c_2w^1, \ldots$$

In this dispersionless limit we again select quadratic relationships (6) for explicit computation of a bi-Hamiltonian structure.

Since roots $v^k$ are conservation law densities, then $w^1 = v^1 + v^2 + v^3$ is again conservation law density (according to the Viète theorem, see (10) and (52)). Since $c_0$ is a conservation law density, then also $w^2$ is a conservation law density. Since (according to the Viète theorem, see (10), (13) and (52))

$$w^2 = -\frac{1}{2}(v^1v^2 + v^1v^3 + v^2v^3),$$

a dispersionless limit of the intermediate system possesses a first local Hamiltonian structure

$$v^i_t = K^{im}\left(\frac{\partial h_2}{\partial v^m}\right)_x,$$

where the Hamiltonian density $h_2 = 2c_1 = w^3 + 2w^1w^2$ (cf. (32)). Thus corresponding hydrodynamic type system takes the symmetric form

$$v^k_t = 2\left(\frac{(v^k)^2}{2} - w^2\right)_x, \quad k = 1, 2, 3. \tag{55}$$

Also (54) can be written in the quadratic form (see (6))

$$c_2 = \frac{1}{2}(w^2)^2 - c_1w^1,$$

where we took into account (53). Thus hydrodynamic type system (55) possesses a second local Hamiltonian structure (cf. (43))

$$w^1_t = \left(\frac{\partial h_1}{\partial c_1}\right)_x, \quad w^2_t = -\left(\frac{\partial h_1}{\partial w^2}\right)_x, \quad c_{1,t} = \left(\frac{\partial h_1}{\partial w^1}\right)_x, \tag{56}$$

where the Hamiltonian density $h_1 = c_3 = -2c_1w^2 + c_1(w^1)^2$:

$$w^1_t = 2\left(\frac{(w^1)^2}{2} - w^2\right)_x, \quad w^2_t = 2c_{1,x}, \quad c_{1,t} = 2\left(c_1w^1\right)_x. \tag{57}$$

Under the substitution $r = \Lambda p + 1$, (44) leads to

$$\Lambda^2(p^3 - p^2w^1 - 2pw^2 - w^3) + \epsilon\Lambda(3pp_x - pw^1_x - w^1p_x - w^2_x) + w^1 - \epsilon w^1_x - p + \epsilon^2 p_{xx} = 0. \tag{58}$$

Then (51) becomes (here $\Lambda^{-2} \to 2\tilde{\lambda}$)

$$\tilde{\lambda} = \frac{p^2}{2} - w^2 - \frac{\frac{1}{2}w^3 + w^1w^2}{p - w^1}. \tag{59}$$

One can verify by straightforward computation that hydrodynamic type system (55) possesses a generating function of conservation laws

$$p_t = 2\left(\frac{p^2}{2} - w^2\right)_x. \tag{60}$$

**Remark**: Algebraic curve of genus zero (59) is nothing but a dispersionless limit of the pseudo-differential operator

$$L = \partial_x^2 - 2w^2 + m\partial_x^{-1}n \tag{61}$$

well known in theory of the Yajima–Oikawa system (see these detail, for instance, in [10]), generating function of conservation law densities (60) is common for any hydrodynamic reduction of the dispersionless KP hierarchy including a dispersionless limit of the Yajima–Oikawa system (see detail in [8]). Thus this hydrodynamic type system (55) is of the Egorov type and moreover is associated with a particular solution

$$z(w^1, c_1) = \frac{1}{6}c_1(w^1)^3 + \frac{3}{4}(c_1)^2 - \frac{1}{2}(c_1)^2 \ln c_1.$$

of another WDVV associativity equation (see detail in [1] and [9]):

$$\frac{\partial^3 z}{\partial c_1 \partial(w^1)^2}\frac{\partial^3 z}{\partial(c_1)^2\partial w^1} = 1 + \frac{\partial^3 z}{\partial(c_1)^3}\frac{\partial^3 z}{\partial(w^1)^3}. \tag{62}$$

## 5.2 An Integrable Dispersive Extension

If $\epsilon \to 0$, but $\epsilon \neq 0$, then intermediate dispersive system can be considered as an integrable dispersive perturbation of hydrodynamic type system (55):

$$\begin{aligned} v_t^1 &= [(v^1+v^2)(v^1+v^3) + \epsilon(...) + \epsilon^2(...) + ...]_x, \\ v_t^2 &= [(v^1+v^2)(v^2+v^3) + \epsilon(...) + \epsilon^2(...) + ...]_x, \\ v_t^3 &= [(v^1+v^3)(v^2+v^3) + \epsilon(...) + \epsilon^2(...) + ...]_x, \end{aligned} \tag{63}$$

which can be written in the Hamiltonian form

$$v_t^i = K^{im}\left(\frac{\delta \mathbf{H}_2}{\delta v^m}\right)_x, \tag{64}$$

where the Hamiltonian density is (see (47), (48))

$$h_2 = (1-\epsilon\partial_x)(2-\epsilon\partial_x)c_1 - \epsilon(c_0 w^1)_x = w^3 + 2w^1(1-\epsilon\partial_x)^{-1}w^2, \tag{65}$$

or in expanded form:

$$h_2 = w^3 + 2w^1w^2 + 2\epsilon w^1 w_x^2 + 2\epsilon^2 w^1 w_{xx}^2 + 2\epsilon^3 w^1 w_{xxx}^2 + ... \tag{66}$$

Taking into account that (49) can be written in the quadratic form

$$(1-\epsilon\partial_x)c_2 = \frac{1}{2}(c_0)^2 - c_1 w^1,$$

the above intermediate dispersive system also can be equipped by another local Hamiltonian structure

$$w_t^1 = \left(\frac{\delta \mathbf{H}_1}{\delta c_1}\right)_x, \quad c_{0,t} = -\left(\frac{\delta \mathbf{H}_1}{\delta c_0}\right)_x, \quad c_{1,t} = \left(\frac{\delta \mathbf{H}_1}{\delta w^1}\right)_x, \tag{67}$$

where the Hamiltonian density is (see (47), (49), (50))

$$h_1 = \frac{1}{2}(1-\epsilon\partial_x)(2-\epsilon\partial_x)c_3 + \frac{3}{2}\epsilon(c_0c_1)_x - \frac{1}{2}\epsilon(c_2w^1)_x$$

$$= \frac{1}{2}(c_0)^2w^1 + 2c_0c_1 + w^1(1-\epsilon\partial_x)^{-1}\left(c_1w^1 - \frac{1}{2}(c_0)^2\right) + \epsilon c_1c_{0,x},$$

or in expanded form:

$$h_1 = 2c_0c_1 + c_1(w^1)^2 + \epsilon c_1c_{0,x} + \epsilon\left(\frac{1}{2}(c_0)^2 - c_1w^1\right)w^1_x$$

$$+\epsilon^2\left(c_1w^1 - \frac{1}{2}(c_0)^2\right)w^1_{xx} + \epsilon^3\left(\frac{1}{2}(c_0)^2 - c_1w^1\right)w^1_{xxx} + ...$$

In flat coordinates $w^1, c_0, c_1$ of second local Hamiltonian structure (67) the intermediate dispersive system is nonlocal:

$$w^1_t = [2c_0 + \epsilon c_{0,x} + w^1(1+\epsilon\partial_x)^{-1}w^1]_x,$$

$$c_{0,t} = [-2c_1 + \epsilon c_{1,x} + c_0w^1 - c_0(1+\epsilon\partial_x)^{-1}w^1]_x,$$

$$c_{1,t} = \left(\frac{1}{2}(c_0)^2 + c_1(1+\epsilon\partial_x)^{-1}w^1 + (1-\epsilon\partial_x)^{-1}\left(c_1w^1 - \frac{1}{2}(c_0)^2\right)\right)_x.$$

If we introduce the field variable $g$ such that

$$w^1 = (1+\epsilon\partial_x)g,$$

then the intermediate dispersive system can be written in a little bit more simple form

$$(1+\epsilon\partial_x)g_t = [2c_0 + \epsilon c_{0,x} + g(g+\epsilon g_x)]_x, \tag{68}$$

$$c_{0,t} = [-2c_1 + \epsilon c_{1,x} - \epsilon c_0g_x]_x,$$

$$(1-\epsilon\partial_x)c_{1,t} = [(2c_1 - \epsilon c_{1,x})g - \epsilon c_0c_{0,x}]_x.$$

Its Hamiltonian structure is

$$(1+\epsilon\partial_x)g_t = \left(\frac{\delta\mathbf{H}_1}{\delta c_1}\right)_x, \quad c_{0,t} = -\left(\frac{\delta\mathbf{H}_1}{\delta c_0}\right)_x, \quad (1-\epsilon\partial_x)c_{1,t} = \left(\frac{\delta\mathbf{H}_1}{\delta g}\right)_x,$$

where

$$h_1 = c_1[2c_0 + \epsilon c_{0,x} + g^2 + \epsilon gg_x] + \frac{1}{2}\epsilon g_x(c_0)^2.$$

A corresponding local Lagrangian representation is

$$S_1 = \int[\frac{1}{2}(G_x - \epsilon^2G_{xxx})Q_t + \frac{1}{2}(Q_x - \epsilon^2Q_{xxx})G_t - \frac{1}{2}F_xF_t - h_1]dxdt,$$

where we introduced functions $Q, F, G$ such that

$$c_1 = (1+\epsilon\partial_x)Q_x, \quad g = G_x, \quad c_0 = F_x.$$

Since any conservation law density is determined up to a total derivative, the Hamiltonian density (66) can be also written in the expanded form

$$h_2 = w^3 + 2w^1w^2 - 2\epsilon w^2 w^1_x + 2\epsilon^2 w^2 w^1_{xx} - 2\epsilon^3 w^2 w^1_{xxx} + ...$$

This means that this Hamiltonian density in a compact form becomes (cf. (65))

$$h_2 = w^3 + 2w^2(1+\epsilon\partial_x)^{-1}w^1.$$

Taking into account again (24), the Hamiltonian density takes the form

$$h_2 = v^1v^2v^3 - (v^1v^2 + v^1v^3 + v^2v^3)(1+\epsilon\partial_x)^{-1}(v^1+v^2+v^3).$$

Then introducing differential substitutions

$$v^k = s^k + \epsilon s^k_x,$$

the Hamiltonian density reduces to the local expression

$$h_2 = (s^1+\epsilon s^1_x)(s^2+\epsilon s^2_x)(s^3+\epsilon s^3_x) \tag{69}$$
$$-(s^1+s^2+s^3)[(s^1+\epsilon s^1_x)(s^2+\epsilon s^2_x) + (s^1+\epsilon s^1_x)(s^3+\epsilon s^3_x) + (s^2+\epsilon s^2_x)(s^3+\epsilon s^3_x)].$$

Then local Hamiltonian structure (64) transforms into nonlocal:

$$s^i_t = K^{im}(1-\epsilon^2\partial^2_x)^{-1}\left(\frac{\delta \mathbf{H}_2}{\delta s^m}\right)_x.$$

A corresponding intermediate dispersive system takes the following non-evolutionary form

$$s^i_t - \epsilon^2 s^i_{xxt} = K^{im}\left(\frac{\delta \mathbf{H}_2}{\delta s^m}\right)_x, \tag{70}$$

where (see (11) and (69)), for instance,

$$K^{1m}\frac{\delta \mathbf{H}_2}{\delta s^m} = (s^1-s^2)(s^1-s^3)$$
$$+\frac{\epsilon}{2}[(-2s^1+s^2+s^3)s^1_x + (s^3-s^1)s^2_x + (s^2-s^1)s^3_x]$$
$$+\frac{\epsilon^2}{2}[-(2s^1+s^2+s^3)s^1_{xx} + (s^1-s^3)s^2_{xx} + (s^1-s^2)s^3_{xx} - 2(s^1_x)^2]$$
$$+\frac{\epsilon^3}{2}[(s^2_x+s^3_x)s^1_{xx} + (s^1_x-s^3_x)s^2_{xx} + (s^1_x-s^2_x)s^3_{xx}],$$

while two other expressions $K^{2m}\frac{\delta \mathbf{H}_2}{\delta s^m}$ and $K^{3m}\frac{\delta \mathbf{H}_2}{\delta s^m}$ can be obtained by a cyclic permutation of indices. Thus the intermediate dispersive system also has a local Lagrangian representation (see (70))

$$S_2 = \int\left(\frac{1}{2}K_{im}(\varphi^i_x - \epsilon^2\varphi^i_{xxx})\varphi^m_t - h_2\right)dxdt,$$

where $\varphi^k$ are potentials such that $s^k=\varphi^k_x$ and $K_{im}$ is determined by (11).

In field variables $w^k$ the first local Hamiltonian operator is precisely (3):

$$\hat{A}_1=\begin{pmatrix} -\frac{3}{2}\partial_x & \frac{1}{2}\partial_x w^1 & \partial_x w^2 \\ \frac{1}{2}w^1\partial_x & \frac{1}{2}(\partial_x w^2+w^2\partial_x) & \frac{3}{2}w^3\partial_x+w^3_x \\ w^2\partial_x & \frac{3}{2}\partial_x w^3-w^3_x & [(w^2)^2-w^1w^3]\partial_x+\partial_x[(w^2)^2-w^1w^3] \end{pmatrix}. \tag{71}$$

One can recalculate a second local Hamiltonian operator in the same coordinates $w^k$, taking into account (47) and differential substitutions (47), (48), (49). In this case a first order local Hamiltonian operator written in field variables $w^1, c_0, c_1$ becomes nonhomogeneous local third order Hamiltonian operator written in field variables $w^k$:

$$w^1_t=(2\partial_x+3\epsilon\partial_x^2+\epsilon^2\partial_x^3)\frac{\delta\mathbf{H}_1}{\delta w^3}, \tag{72}$$

$$w^2_t=-\partial_x\frac{\delta\mathbf{H}_1}{\delta w^2}+2(w^1_x+w^1\partial_x)\frac{\delta\mathbf{H}_1}{\delta w^3}+\epsilon\left(-2w^1_{xx}-3w^1_x\partial_x-w^1\partial_x^2\right)\frac{\delta\mathbf{H}_1}{\delta w^3}$$
$$+\epsilon^2\left(\partial_x^3\frac{\delta\mathbf{H}_1}{\delta w^2}-(w^1_{xx}\partial_x+2w^1_x\partial_x^2+w^1\partial_x^3)\frac{\delta\mathbf{H}_1}{\delta w^3}\right),$$

$$w^3_t=2\partial_x\frac{\delta\mathbf{H}_1}{\delta w^1}+2w^1\partial_x\frac{\delta\mathbf{H}_1}{\delta w^2}-(4w^1w^1_x+4w^2_x+(8w^2+4(w^1)^2)\partial_x)\frac{\delta\mathbf{H}_1}{\delta w^3}$$
$$+\epsilon\left(-3\partial_x^2\frac{\delta\mathbf{H}_1}{\delta w^1}+(w^1\partial_x^2-w^1_x\partial_x)\frac{\delta\mathbf{H}_1}{\delta w^2}+(2w^1w^1_{xx}+2w^2_{xx}+2(w^1_x)^2+(4w^1w^1_x+4w^2_x)\partial_x)\frac{\delta\mathbf{H}_1}{\delta w^3}\right)$$
$$+\epsilon^2\left(\partial_x^3\frac{\delta\mathbf{H}_1}{\delta w^1}-(w^1_x\partial_x^2+w^1\partial_x^3)\frac{\delta\mathbf{H}_1}{\delta w^2}\right)$$
$$+\epsilon^2[(w^2_{xx}+w^1w^1_{xx}+(w^1_x)^2)\partial_x+(3w^1w^1_x+3w^2_x)\partial_x^2+((w^1)^2+2w^2)\partial_x^3]\frac{\delta\mathbf{H}_1}{\delta w^3}.$$

Intermediate dispersive system (63) in field variables $w^k$ takes the form (see (71))

$$w^1_t=[(w^1)^2-2w^2-\epsilon(w^1w^1_x+3w^2_x)+\epsilon^2(w^1w^1_{xx}-3w^2_{xx})-\epsilon^3(w^1w^1_{xxx}+3w^2_{xxx})+...]_x,$$
$$w^2_t=[2w^1w^2+w^3+\epsilon(w^1w^2_x-2w^2w^1_x)+\epsilon^2(w^1w^2_{xx}-w^1_xw^2_x+2w^2w^1_{xx})$$
$$+\epsilon^3(w^1w^2_{xxx}+w^2_xw^1_{xx}-w^1_xw^2_{xx}-2w^2w^1_{xxx})+...]_x,$$
$$w^3_t=4w^2w^2_x+2w^3w^1_x+\epsilon(2w^2w^2_{xx}-w^1_xw^3_x-3w^3w^1_{xx})$$
$$+\epsilon^2(2w^2w^2_{xxx}+w^3_xw^1_{xx}+3w^3w^1_{xxx})+\epsilon^3(2w^2w^2_{xxxx}-w^3_xw^1_{xxx}-3w^3w^1_{xxxx})+...,$$

where the Hamiltonian density

$$h_1=(w^1)^3w^2-2w^1(w^2)^2+\frac{1}{2}(w^1)^2w^3-w^2w^3+\epsilon\left(2(w^2)^2w^1_x+2(w^1)^3w^2_x+(w^1)^2w^3_x\right)+...$$

# 6 The Yajima–Oikawa System and its Dispersionless Limit

The remarkable Yajima–Oikawa system is associated with linear equation (61):

$$\psi_{xx} - 2w^2\psi + m\partial_x^{-1}n\psi = \lambda\psi,$$

which can be written as a third order linear ordinary differential equation

$$\psi_{xxx} - \frac{m_x}{m}\psi_{xx} - 2w^2\psi_x + \left(-2w_x^2 + 2w^2\frac{m_x}{m}\right)\psi + nm\psi = \lambda\left(\psi_x - \frac{m_x}{m}\psi\right)$$

or in the factorized form

$$(\partial_x - v^1)(\partial_x - v^2)(\partial_x - v^3)\psi = \lambda(\partial_x - w^1)\psi.$$

Substitution (18) leads to the nonlinear ordinary differential equation of second order

$$r_{xx} + 3rr_x + r^3 - w^1(r_x + r^2) - 2w^2r - w^3 = \lambda(r - w^1),$$

where we introduced new field variables $w^1 = (\ln m)_x, w^3 = 2w_x^2 - 2w^1w^2 - nm$.

Now we incorporate a parameter $\epsilon$ changing $\partial_x \to \epsilon\partial_x$, and consider the equation

$$\epsilon^2 r_{xx} + 3\epsilon rr_x + r^3 - w^1(\epsilon r_x + r^2) - 2w^2r - w^3 = \lambda(r - w^1),$$

whose unknown functions $w^k$ can be expressed via roots $v^k$ (cf. (24)):

$$w^1 = v^1 + v^2 + v^3, \quad -2w^2 = v^1v^2 + v^1v^3 + v^2v^3 - \epsilon(v_x^2 + 2v_x^3),$$

$$w^3 = v^1v^2v^3 - \epsilon(v^1v_x^3 + v^2v_x^3 + v^3v_x^2) + \epsilon^2 v_{xx}^3.$$

Below we present main results of this Section and omit all computations, because they are similar to the previous Sections.

**Proposition**: *The evolutionary system* (cf. (57))

$$w_t^1 = 2\left(\frac{(w^1)^2}{2} - w^2 + \frac{\epsilon}{2}w_x^1\right)_x, \quad w_t^2 = 2c_{1,x}, \quad c_{1,t} = 2\left(c_1w^1 - \frac{\epsilon}{2}c_{1,x}\right)_x \tag{73}$$

*is a bi-Hamiltonian system.*

Indeed $w^1, w^2, c_1$ are flat coordinates of the second local Hamiltonian structure, i.e. (cf. (56))

$$w_t^1 = \left(\frac{\delta\mathbf{H}_1}{\delta c_1}\right)_x, \quad w_t^2 = -\left(\frac{\delta\mathbf{H}_1}{\delta w^2}\right)_x, \quad c_{1,t} = \left(\frac{\delta\mathbf{H}_1}{\delta w^1}\right)_x,$$

where the Hamiltonian density is $h_1 = c_1[(w^1)^2 - 2w^2 + \epsilon w_x^1]$, and $c_1 = \frac{1}{2}w^3 + w^1w^2 - \epsilon w_x^2$ is a conservation law density.

In flat coordinates $v^k$ of the first local Hamiltonian structure

$$v_t^i = K^{im}\left(\frac{\delta\mathbf{H}_2}{\delta v^m}\right)_x$$

evolutionary system (73) takes the form

$$\begin{aligned} v_t^1 &= [(v^1+v^2)(v^1+v^3)+\epsilon(v^1+v^2)_x]_x, \\ v_t^2 &= [(v^1+v^2)(v^2+v^3)]_x, \\ v_t^3 &= [(v^1+v^3)(v^2+v^3)-\epsilon(v^2+v^3)_x]_x, \end{aligned}$$

where the Hamiltonian density is

$$h_2 = 2w^1w^2 + w^3$$

$$= v^1v^2v^3 - (v^1+v^2+v^3)(v^1v^2+v^1v^3+v^2v^3) + \epsilon(v^1v_x^2 + (v^1+v^2)v_x^3).$$

# 7 Another Dispersionful Version of the WDVV associativity System

In this paper we dealt with the first "half" of linear spectral problem (9). However completely the same set of computations can be made for the second "half" of this linear spectral problem (9). Indeed the matrix system

$$\boldsymbol{\psi}_y = \lambda \begin{pmatrix} 0 & 0 & 1 \\ a^3 & a^2 & 0 \\ (a^2)^2 - a^1a^3 & a^3 & 0 \end{pmatrix} \boldsymbol{\psi}$$

can be written as a single ordinary differential equation of third order

$$\psi_{yyy} - \left(\lambda a^2 + \frac{a_y^3}{a^3}\right)\psi_{yy} + \lambda^2[a^1a^3 - (a^2)^2]\psi_y + \lambda^2\left(a^3a_y^1 - 2a^2a_y^2 + \frac{(a^2)^2}{a^3}a_y^3 + \lambda[(a^2)^3 - a^1a^2a^3 - (a^3)^2]\right)\psi = 0,$$

where

$$\psi_2 = \frac{1}{\lambda}\psi_y, \quad \psi_1 = \frac{1}{\lambda^2 a^3}\psi_{yy} + \left(a^1 - \frac{(a^2)^2}{a^3}\right)\psi.$$

Under the scaling transformation (cf. (15))

$$y = \epsilon(e^{t/\epsilon} - 1), \tag{74}$$

this third order equation becomes

$$\psi_{ttt} - \left(\lambda w^2 + \frac{3}{2\epsilon} + \frac{w_t^3}{w^3}\right)\psi_{tt} + \left(\lambda^2[w^1w^3 - (w^2)^2] + \frac{1}{\epsilon}\lambda w^2 + \frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\frac{w_t^3}{w^3}\right)\psi_t$$

$$+\lambda^2\left(w^3w_t^1 - 2w^2w_t^2 + \frac{(w^2)^2}{w^3}w_t^3 - \frac{1}{2\epsilon}w^1w^3 + \frac{1}{2\epsilon}(w^2)^2 + \lambda[(w^2)^3 - w^1w^2w^3 - (w^3)^2]\right)\psi = 0,$$

where we introduced new field variables $w^k$ such that

$$a^1 = w^1e^{-t/(2\epsilon)}, \quad a^2 = w^2e^{-t/\epsilon}, \quad a^3 = w^3e^{-3t/(2\epsilon)}.$$

In a high-frequency limit corresponding WDVV associativity system also has a bi-Hamiltonian structure (see detail in [6]). One can completely repeat the computations made above in this paper and derive another dispersionful version of an intermediate system whose high-frequency limit must coincide with original WDVV associativity system up to change of independent variables ($y \leftrightarrow z$).

$$\epsilon^2 r_{tt} + 3\epsilon r r_t + r^3 - \left(\Lambda w^2 + \frac{3}{2} + \epsilon\frac{w_t^3}{w^3}\right)(\epsilon r_t + r^2) + \left(\Lambda^2[w^1 w^3 - (w^2)^2] + \Lambda w^2 + \frac{1}{2} + \epsilon\frac{w_t^3}{w^3}\right) r$$

$$+\Lambda^2\left[\frac{1}{2}(w^2)^2 - \frac{1}{2}w^1 w^3 + \epsilon\left(w^3 w_t^1 - 2w^2 w_t^2 + \frac{(w^2)^2}{w^3} w_t^3\right)\right] + \Lambda^3[(w^2)^3 - w^1 w^2 w^3 - (w^3)^2] = 0,$$

The substitution $r = \Lambda p + \frac{1}{2}$ yields (cf. (58))

$$\Lambda^3[p^3 - w^2 p^2 + [w^1 w^3 - (w^2)^2]p + (w^2)^3 - w^1 w^2 w^3 - (w^3)^2]$$

$$+\epsilon\Lambda^2\left[w^3 w_t^1 - 2w^2 w_t^2 + \frac{(w^2)^2}{w^3} w_t^3 + 3p p_t - w^2 p_t - \frac{w_t^3}{w^3} p^2\right]$$

$$+\Lambda\left(\frac{1}{4}w^2 - \frac{1}{4}p + \epsilon^2 p_{tt} - \epsilon^2\frac{w_t^3}{w^3} p_t\right) + \frac{1}{4}\epsilon\frac{w_t^3}{w^3} = 0.$$

Then in a dispersionless limit one can obtain again (cf. (59))

$$\bar{\lambda} = p^2 + w^1 w^3 - (w^2)^2 - \frac{(w^3)^2}{p - w^2},$$

where $\bar{\lambda} = (2\Lambda)^{-2}$. This means that this dispersionless limit also is associated with the Yajima–Oikawa hierarchy.

# 8 Conclusion

In this paper we investigated the intermediate dispersive system, which possesses two limits:

1. a high-frequency limit, nondiagonalizable hydrodynamic type system (12)

$$u_y^1 = \frac{1}{2}(u^2 u^3 - u^1 u^2 - u^1 u^3)_z, \quad u_y^2 = \frac{1}{2}(u^1 u^3 - u^1 u^2 - u^2 u^3)_z, \quad u_y^3 = \frac{1}{2}(u^1 u^2 - u^1 u^3 - u^2 u^3)_z$$

integrable by the inverse scattering transform;

2. a dispersionless limit, semi-Hamiltonian hydrodynamic type system (55)

$$v_t^1 = [(v^1 + v^2)(v^1 + v^3)]_x, \quad v_t^2 = [(v^1 + v^2)(v^2 + v^3)]_x, \quad v_t^3 = [(v^1 + v^3)(v^2 + v^3)]_x$$

integrable by the generalized hodograph method (see detail in [11]).

Introducing the potential function $F$ such that (see (57))

$$w^2 = F_{xx}, \quad c_1 = \frac{1}{2}F_{xt}, \quad w^1 = \frac{F_{tt}}{2F_{xt}}$$

this semi-Hamiltonian hydrodynamic type system can be written as a single equation of third order (cf. (2))

$$F_{ttt} = 2\frac{F_{tt}}{F_{xt}}F_{xtt} - \frac{F_{tt}^2}{F_{xt}^2}F_{xxt} - 4F_{xt}F_{xxx}.$$

The WDVV associativity system (12) admits two different dispersionful extensions based on simple transformations (15) and (74). Their dispersionless limits coincide with a dispersionless limit of the Yajima–Oikawa system (55).

The intermediate dispersive system (see (35), here we denote $\xi = \epsilon^{-1}$)

$$(\partial_x + \xi)\eta_t = (q_{0,x} + \xi(2q_0 + \eta\eta_x) + \xi^2\eta^2)_x, \quad q_{0,t} = (q_{1,x} - q_0\eta_x - 2\xi q_1)_x, \tag{75}$$

$$(\partial_x - \xi)q_{1,t} = (q_0 q_{0,x} + \xi\eta q_{1,x} - 2\xi^2\eta q_1)_x$$

reduces to original WDVV associativity equation (2) in a high-frequency limit ($\xi \to 0$), where $\eta = f_{xx}, q_0 = f_{xt}$ and the function $q_1$ is determined by its first derivatives (the compatibility condition $(q_{1,x})_t = (q_{1,t})_x$ yields again (2)):

$$q_{1,x} = f_{tt} + f_{xt}f_{xxx}, \quad q_{1,t} = f_{xt}f_{xxt}.$$

The intermediate dispersive system is bi-Hamiltonian and both corresponding Lagrangian representations (34) and (36) are local.

In Dubrovin's approach (the so-called integrable higher dispersive corrections) theory of WDVV associativity equations plays a very important role. Any solution of (2) leads to two three-component commuting hydrodynamic type systems (see detail in [1] and [9]), which are Hamiltonian and Egorov (see detail in [10]). Integrable dispersive corrections of these hydrodynamic type systems simultaneously allow to construct integrable dispersive correction for WDVV associativity system (1). In our paper we used an alternative strategy: we constructed an integrable perturbation (75). If $\xi \to 0$ (i.e. a high-frequency limit), we came back to (1); if $\xi \to \infty$ (i.e. a dispersionless limit), we obtained a single three-component hydrodynamic type system (55), which is bi-Hamiltonian and Egorov. This system is determined by a particular solution of another WDVV associativity equation (see (62) and cf. (2))

$$f_{xxt}f_{xtt} = 1 + f_{xxx}f_{ttt}.$$

Both WDVV associativity equations are connected with each other by a special reciprocal transformation (see [5]). If $\xi$ is arbitrary (assume for simplicity that $\xi$ is small), then intermediate dispersive system (75) can be interpreted as an integrable system describing integrable corrections (in any order with respect to $\xi$) of Hamiltonian and Egorov three-component hydrodynamic type systems. However these integrable corrections should have another interpretation in comparison with Dubrovin's approach. This separate investigation should be made somewhere else.

If parameter $\epsilon$ is small, intermediate dispersive system (68) has a pair of Hamiltonian structures (71), (72), which become first order Hamiltonian structures of Dubrovin–Novikov type (see [2]) in a dispersionless limit. If parameter $\xi$ is small, intermediate dispersive system (75) has the same pair of Hamiltonian structures, but the second Hamiltonian operator reduces to third order Hamiltonian operator of Dubrovin–Novikov type (see [3]) in a high-frequency limit (see again [4]).

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